TOPOLOGIZATION OF SETS ENDOWED WITH AN ACTION OF A MONOID

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ABSTRACT. Given a set X and a family G of functions from X to X we pose and explore the question of the existence of a non-discrete Hausdorff topology on X such that all functions $f \in G$ are continuous. A topology on X with the latter property is called a *G-topology*. The answer will be given in terms of the Zariski G-topology ζ_G on X. This is the topology generated by the subbase consisting of the sets $\{x \in X : f(x) \neq g(x)\}$ and $\{x \in X : f(x) \neq c\}$ where $f, g \in G$, $c \in X$. We prove that for a countable submonoid $G \subset X^X$ the G-act X admits a non-discrete Hausdorff G-topology if and only if the Zariski G-topology ζ_G is not discrete if and only if X admits $2^{\mathfrak c}$ normal G-topologies.

1. Principal Problems

In this paper we consider the following general problem:

Problem 1.1. *Given a set X and a family G of functions from X to X detect if X admits a non-discrete Hausdorff (or normal) topology such that all functions $g \in G$ are continuous.*

Since the composition of continuous functions is continuous, we lose no generality assuming that the family G is a subsemigroup of the semigroup X^X of all functions $X \to X$, endowed with the operation of composition. Also we can assume that G contains the identity function id_X of X and hence G is a submonoid of X^X. Thus it is natural to consider Problem 1.1 in the context of *G-acts*, i.e. sets endowed with an action of a monoid G [7]. The two-sided unit of the monoid G will be denoted by 1_G. An *action* of a monoid G on a set X is a function $\alpha : G \times X \to X$, $\alpha : (g, x) \mapsto g(x)$ that has two properties:

- $1_G(x) = x$ for all $x \in X$ and
- $f(g(x)) = (fg)(x)$ for all $f, g \in G$ and $x \in X$.

A topology τ on a G-act X is called a *G-topology* if for every $g \in G$ the shift $g : X \to X$, $g : x \mapsto g(x)$, is continuous. A G-act X is called (*normally*) *G-topologizable* if X admits a (normal) Hausdorff G-topology. A topology τ on a set X is called *normal* if the topological space (X, τ) is normal in the sense that X is a T_1-space such that any two disjoint closed subsets in X have disjoint open neighborhoods.

In this terminology Problem 1.1 can be rewritten as follows.

Problem 1.2. *Find necessary and sufficient conditions of (normal) G-topologizability of a given G-act X.*

For G-acts endowed with an action of a group G this problem has been considered in [1]. For countable monoids G, Problem 1.2 will be answered in Theorem 5.4 proved in Section 5. The answer will be given in terms of the Zariski G-topology on X, defined and studied in Section 2. In Section 3 we investigate largest G-topologies, generated by (special) filters.

2. The Zariski G-topology on a G-act

In this section we define the Zariski G-topology on a G-act X and study this topology on some concrete examples of G-acts.

Definition 2.1. For a monoid G and a G-act X the *Zariski G-topology* ζ_G on X is the topology generated by the subbase $\tilde{\zeta}_G$ consisting of the sets $\{x \in X : f(x) \neq g(x)\}$ and $\{x \in X : f(x) \neq c\}$ where $f, g \in G$ and $c \in X$.

The following easy fact follows immediately from the definition.

Proposition 2.2. *For any G-act X the Zariski G-topology ζ_G satisfies the separation axiom T_1 and lies in any Hausdorff G-topology on X.*

Now let us introduce a cardinal characteristic $\psi(x,\tilde\zeta_G)$ of the subbase $\tilde\zeta_G$ of the Zariski G-topology ζ_G called the pseudocharacter of $\tilde\zeta_G$ at a point $x\in X$. In fact, the pseudocharacter $\psi(x,\mathcal F)$ can be defined for any family $\mathcal F$ of subsets of X. Given a point $x\in X$ let

$$\mathcal F(x)=\{X\}\cup\{F\in\mathcal F: x\in F\}$$

and define the *pseudocharacter* of $\mathcal F$ at x as

$$\psi(x,\mathcal F)=\min\{|\mathcal U|:\mathcal U\subset\mathcal F(x)\ \text{ and }\ \cap\mathcal U=\cap\mathcal F(x)\}.$$

It τ is the topology on X generated by the subbase $\mathcal F$, then $\tau(x)$ is the family of all open neighborhoods of x and $\psi(x,\tau)$ is the usual pseudocharacter of the point x in the topological space (X,τ). It is easy to see that $\psi(x,\tau)=\psi(x,\mathcal F)$ for any non-isolated point x in (X,τ). If x is isolated in (X,τ), then $\psi(x,\tau)=1$ while $1\le\psi(x,\mathcal F)<\aleph_0$, so the pseudocharacter $\psi(x,\tilde\zeta_G)$ carries more information than $\psi(x,\zeta_G)$ in case of an isolated point x in (X,ζ_G). If x is non-isolated, then $\psi(x,\tilde\zeta_G)=\psi(x,\zeta_G)$.

In the algebraic language the pseudocharacter $\psi(x,\tilde\zeta_G)$ equals the smallest number of inequalities of the form

$$f(x)\ne g(x)\ \text{ or }\ f(x)\ne c\ \text{ where }\ f,g\in G,\ c\in X$$

in a system of inequalities whose unique solution is x.

Now let us consider the Zariski G-topology on some concrete examples of G-acts.

Example 2.3. Let X be an infinite set endowed with the natural action of the group G of all bijective functions $f:X\to X$ that have finite support

$$\operatorname{supp}(f)=\{x\in X: f(x)\ne x\}.$$

It is easy to see that $\psi(x,\zeta_G)=1$ and $\psi(x,\tilde\zeta_G)=2$ for any point $x\in X$. Consequently, the Zariski G-topology ζ_G on X is discrete and the G-act X is not G-topologizable.

Each group G can be considered as an S-act for many natural actions of various submonoids S of the monoid G^G. We define 6 such natural submonoids of G^G:

- G_l is the subgroup of G^G that contains all left shifts $l_a:x\mapsto ax$ of G for $a\in G$;
- G_r is the subgroup of G^G that contains all right shifts $r_a:x\mapsto xa$ of G for $a\in G$;
- G_s is the subgroup of G^G that contains all two-sided shifts $s_{a,b}:x\mapsto axb$ of G for $a,b\in G$;
- G_q is the subgroup of G^G that contains all bijections of the form $f:x\mapsto ax^\varepsilon b$ where $a,b\in G$ and $\varepsilon\in\{1,-1\}$;
- G_m is the smallest submonoid of G^G that contains all functions of the form $f:x\mapsto ax^mb$ where $a,b\in G$ and $m\in\mathbb Z$;
- G_p is the smallest submonoid that contains the subgroup G_q and together with any two functions $f,g\in G_p$ contains their product $f\cdot g:x\mapsto f(x)\cdot g(x)$.

Functions from the families G_m and G_p will be called *monomials* and *polynomials* on the group G, respectively.

It is clear that

$$G_l\cup G_r\subset G_s\subset G_q\subset G_m\subset G_p$$

and hence

$$\zeta_{G_l}\cup\zeta_{G_r}\subset\zeta_{G_s}\subset\zeta_{G_q}\subset\zeta_{G_m}\subset\zeta_{G_p}.$$

A group G endowed with a G_l-topology (resp. G_r-topology, G_s-topology, G_q-topology) is called *left-topological* (resp. *right-topological*, *semi-topological*, *quasi-topological*).

Now for each monoid $S\in\{G_l,G_r,G_s,G_q,G_m,G_p\}$ we shall analyse the structure of the Zariski S-topology on the group G. By the *cofinite topology* on a set X we understand the topology

$$\tau_1=\{\emptyset\}\cup\{X\setminus F: F\text{ is a finite subset of }X\}.$$

The following remark can be easily derived from the definitions.

Remark 2.4. For any group G the Zariski topologies ζ_{G_l} and ζ_{G_r} on G coincide with the cofinite topology on G. If G is infinite, then the topologies ζ_{G_l} and ζ_{G_r} are not Hausdorff.

Remark 2.5. For any infinite group G the Zariski topologies ζ_{G_s} and ζ_{G_q} are not discrete. This follows from a deep result of Y.Zelenyuk [14], [15] who proved that each infinite group G admits a non-discrete Hausdorff topology with continuous two-sided shifts and continuous inversion.

Remark 2.6. There is a countable infinite group G whose Zariski G_m-topology ζ_{G_m} is discrete, see [12, p.70].

Remark 2.7. For a group G endowed with the natural action of the monoid G_p of all polynomial functions on G, the Zariski G_p-topology ζ_{G_p} coincides with the usual Zariski topology on the group G, studied in [2], [13], [3]. By the classical Markov's result [9], a countable group G is topologizable (which means that G admits a non-discrete Hausdorff topology that turns G into a topological group) if and only if the Zariski topology ζ_{G_p} is not discrete. Countable non-topologiable groups where constructed in [10] and [8]. For such groups G the Zariski G_p-topology ζ_{G_p} is discrete.

Remark 2.8. The Markov's topologizability criterion is not valid for uncountable groups: in [4] Hesse constructed an example of an uncountable non-topologizable group G whose Zariski topology ζ_{G_p} is discrete.

Therefore, for an infinite group G, the Zariski G_q-topology ζ_{G_q} is always not discrete while the topology ζ_{G_m} can be discrete (for some countable non-topologizable groups).

If a group G is abelian, then the Zariski topology ζ_{G_s} on G coincides with the topologies ζ_{G_l} and ζ_{G_r} and hence is cofinite. However, for non-abelian groups G the topology ζ_{G_s} can have rather unexpected properties.

Example 2.9 (Dikranjan-Toller)**.** Let H be a finite discrete topological group with trivial center (for example, let $H = \Sigma_3$ be the group of bijections of a 3-element set). For any cardinal κ the Zariski topologies ζ_{G_s}, ζ_{G_q} and ζ_{G_p} on the group $G = H^\kappa$ coincide with the Tychonoff product topology τ on $G = H^\kappa$ and hence are compact, Hausdorff, and have pseudocharacter $\psi(x,\tau) = \kappa < 2^\kappa = |G_s| = |G_r| = |G_p| = |G|$ at each point $x \in G$.

Proof. Observe that the Tychonoff product topology τ on $G = H^\kappa$ turns the group G into a compact topological group. Then each polynomial map on G is continuous and each set $U \in \tilde{\zeta}_{G_p}$ is open in X. Consequently, $\zeta_{G_s} \subset \zeta_{G_q} \subset \zeta_{G_p} \subset \tau$. The Tychonoff product topology τ is generated by the subbasic sets

$$U_{\alpha,h} = \{x \in X : \mathrm{pr}_\alpha(x) = h\}$$

where $\alpha \in \kappa$, $h \in H$ and $\mathrm{pr}_\alpha : H^\kappa \to H$ denotes the αth coordinate projection. To prove that $\zeta_{G_s} = \zeta_{G_q} = \zeta_{G_p} = \tau$ it suffices to check that each set $U_{\alpha,h}$ belongs to the topology ζ_{G_s}.

Consider the embedding $i_\alpha : H \to H^\kappa$ that assigns to each element $x \in H$ the point $i_\alpha(x) \in H^\kappa$ such that $\mathrm{pr}_\alpha \circ i_\alpha(x) = x$ and $\mathrm{pr}_\beta \circ i_\alpha(x) = 1_H$ for all $\beta \neq \alpha$.

Given a point $h \in H$, consider the finite set $A_h = \{(a,b) \in H \times H : ah \neq hb\}$ and observe that $\{h\} = \bigcap_{(a,b)\in A_h}\{x \in H : x^{-1}ax \neq b\}$. Indeed, by the triviality of the center of H, for any $x \in H \setminus \{h\}$ there is an element $a \in H$ such that $(xh^{-1})a \neq a(xh^{-1})$ and hence $h^{-1}ah \neq x^{-1}ax$. Put $b = x^{-1}ax$ and observe that $h^{-1}ah \neq b$ and hence $(a,b) \in A_h$.

For each pair $(a,b) \in A_h$ consider the left and right shifts $l_a : x \mapsto i_\alpha(a)\cdot x$ and $r_b : x \mapsto x \cdot i_\alpha(b)$ of the group $G = H^\kappa$. These shifts generate the subbasic set

$$U_{a,b} = \{x \in X : i_\alpha(a)\cdot x \neq x \cdot i_\alpha(b)\} = \{x \in X : l_a(x) \neq r_b(x)\} \in \tilde{\zeta}_{G_s}.$$

It remains to observe that

$$\mathrm{pr}_\alpha^{-1}(h) = \bigcap_{(a,b)\in A_h} U_{a,b} \in \zeta_{G_s},$$

witnessing that $\tau = \zeta_{G_p} = \zeta_{G_q} = \zeta_{G_s}$. □

Remark 2.10. In fact, the Zariski topologies $\zeta_{G_l}, \zeta_{G_r}, \zeta_{G_s}$ can be defined on each semigroup G. If G is commutative, then these Zariski topologies coincide. For the monoid $G = (\mathbb{N}, \max)$ the Zariski topology ζ_{G_s} is discrete. Indeed, for each $n \in \mathbb{N}$, the singleton $\{n\}$ belongs to the topology ζ_{G_s} as

$$\{n\} = \{x \in \mathbb{N} : \max\{x,n\} \neq n\} \cap \bigcap_{k<n}\{x \in \mathbb{N} : x \neq k\}.$$

This implies that the monoid $G = (\mathbb{N}, \max)$ is not G_s-topologizable.

3. G-topologies on G-acts, generated by special filters

In this section we describe and study G-topologies on G-acts, generated by filters.

A *filter* on a set X is a family φ of subsets of X such that

- $\emptyset \notin \varphi$;
- $A \cap B \in \varphi$ for any sets $A, B \in \varphi$;
- $A \cup B \in \varphi$ for any sets $A \in \varphi$ and $B \subset X$.

By the *pseudocharacter* $\psi(\varphi)$ *of a filter* φ we understand the smallest cardinality $|\mathcal{F}|$ of a subfamily $\mathcal{F} \subset \varphi$ such that $\cap\mathcal{F} = \cap\varphi$. The *character* $\chi(\varphi)$ of a filter φ equals the smallest cardinality of a subfamily $\mathcal{F} \subset \varphi$ such that each set $\Phi \in \varphi$ contain some set $F \in \mathcal{F}$. Observe that the *character* $\chi(x, \tau)$ of a topological space (X, τ) at a point x can be defined as the character $\chi(\tau_x)$ of the neighborhood filter $\tau_x = \{U \in \tau : x \in U\}$.

For a filter φ on X consider the family

$$\varphi^+ = \{E \subset X : \forall F \in \varphi \;\; F \cap E \neq \emptyset\}$$

equal to the union of all filters on X that contain φ. It is easy to check that for each $A \subset X$ with $A \notin \varphi$, we get $X \setminus A \in \varphi^+$.

We shall say that a filter φ on a topological space X *converges to a point* x_0 if each neighborhood $U \subset X$ of x_0 belongs to the filter φ.

Now assume that G is a monoid, X is a G-act and φ is a filter on X such that $\bigcap\varphi = \{x_0\}$ for some point x_0. Then we can consider the largest G-topology τ_φ on X for which the filter φ converges to x_0. This topology admits the following simple description:

Proposition 3.1. *The topology τ_φ consists of all sets $U \subset X$ such that for any $g \in G$ with $x_0 \in g^{-1}(U)$ the preimage $g^{-1}(U)$ belongs to the filter φ.*

Now our strategy is to detect filters φ on X generating "nice" G-topology τ_φ on X.

Definition 3.2. Let κ be a cardinal. An injective transfinite sequence $(x_\alpha)_{\alpha<\kappa}$ of points of a G-act X is called *special* if there is an enumeration $G = \{g_\alpha\}_{\alpha<\kappa}$ of the monoid G such that for all ordinals $\alpha < \kappa$ and $\beta, \gamma, \delta < \alpha$ we get

(1) if $g_\beta(x_0) \neq g_\gamma(x_0)$, then $g_\beta(x_\alpha) \neq g_\gamma(x_\alpha)$;
(2) if $g_\beta(x_0) \neq g_\gamma(x_\delta)$, then $g_\beta(x_\alpha) \neq g_\gamma(x_\delta)$.

Definition 3.3. A filter φ on a G-act X is called *special* if for some cardinal κ there is a special sequence $(x_\alpha)_{\alpha<\kappa}$ in X such that $\bigcap\varphi = \{x_0\}$ and $\{x_0\} \cup \{x_\beta : \beta > \alpha\} \in \varphi$ for all ordinals $\alpha < \kappa$. In this case the set $X_0 = \{x_\alpha\}_{\alpha<\kappa}$ is called the *special support* of φ.

For a special filter φ on X the G-topology τ_φ has many nice properties.

Theorem 3.4. *For any special filter φ on a G-act X with special support X_0 and the intersection $\cap\varphi = \{x_0\}$, the G-topology τ_φ has the following properties:*

(1) *the topological space (X, τ_φ) is normal;*
(2) *for any set $F \in \varphi$ the set $G(F) = \{g(x) : g \in G,\ x \in F\}$ is closed-and-open in (X, τ_φ) and $X \setminus G(F)$ is discrete in (X, τ_φ);*
(3) *$\{F \cap X_0 : F \in \varphi\} = \{U \cap X_0 : x_0 \in U \in \tau_\varphi\}$;*
(4) *$\psi(x_0, \tau_\varphi) = \psi(\varphi)$ and $\chi(x_0, \tau_\varphi) \geq \chi(\varphi)$.*

Proof. By definition, the special support X_0 of φ admits an enumeration $X_0 = \{x_\alpha\}_{\alpha<\kappa}$ that has the properties (1), (2) from Definition 3.2 for some enumeration $G = \{g_\alpha\}_{\alpha<\kappa}$ of the monoid G. For every ordinal $\alpha < \kappa$ consider the set $X_{>\alpha} = \{x_\beta : \alpha < \beta < \kappa\}$ and observe that $\{x_0\} \cup X_{>\alpha} \in \varphi$ according to Definition 3.3. Now we shall prove the required properties of the G-topology τ_φ.

Claim 3.5. *The topology τ_φ satisfies the separation axiom T_1.*

Proof. Given any point $x \in X$, we need to show that $X \setminus \{x\} \in \tau_\varphi$. Since the special filter φ contains the sets $\{x_0\} \cup X_{>\alpha}$, $\alpha \in \kappa$, it suffices for every map $g \in G$ with $g(x_0) \in X \setminus \{x\}$ to find $\alpha < \kappa$ such that $g(X_{>\alpha}) \subset X \setminus \{x\}$. If $x \notin G(X_0)$, then $g(X_{>0}) \subset G(X_0) \subset X \setminus \{x\}$ and we are done.

So, assume that $x \in G(X_0)$ and find ordinals $\gamma, \delta < \kappa$ such that $x = g_\gamma(x_\delta)$. Also find an ordinal $\beta < \kappa$ such that $g_\beta = g$. Since $g_\beta(x_0) = g(x_0) \neq x = g_\gamma(x_\delta)$, the condition (2) of Definition 3.2 guarantees that $g(x_\alpha) = g_\beta(x_\alpha) \neq g_\gamma(x_\delta) = x$ for all $\alpha > \max\{\beta, \gamma, \delta\}$. Consequently, for the ordinal $\alpha = \max\{\beta, \gamma, \delta\}$ we get the required inclusion $g(X_{>\alpha}) \subset X \setminus \{x\}$. □

Claim 3.6. *The topology τ_φ is normal.*

Proof. Let A_0, B_0 be two disjoint closed subsets in the topological space (X, τ_φ). Consider the sequences of sets $(A_n)_{n\in\omega}$ and $(B_n)_{n\in\omega}$ defined by the recursive formulas:

$$A_{n+1} = A_n \cup \{g_\alpha(x_\gamma) : \alpha < \gamma < \kappa,\;\; g_\alpha(x_0) \in A_n,\ g_\alpha(x_\gamma) \notin B_n\}$$

and
$$B_{n+1} = B_n \cup \{g_\beta(x_\delta) : \beta < \delta < \kappa,\ \ g_\beta(x_0) \in B_n,\ g_\beta(x_\delta) \notin A_n\}.$$

We claim that the sets $A_\omega = \bigcup_{n\in\omega} A_n$ and $B_\omega = \bigcup_{n\in\omega} B_n$ are open disjoint neighborhoods of the sets A_0 and B_0 in (X,τ_φ). First we check that these sets are disjoint. Assuming the opposite, we can find numbers $n,m\in\omega$ such that $A_{n+1}\cap B_{m+1}\neq\emptyset$ but $A_n\cap B_{m+1}=\emptyset=A_{n+1}\cap B_m$. Choose any point $c\in A_{n+1}\cap B_{m+1}$. By the definitions of the sets A_{n+1} and B_{m+1}, the point c is of the form $g_\alpha(x_\gamma)=c=g_\beta(x_\delta)$ for some ordinals $\alpha<\gamma<\kappa$ and $\beta<\delta<\kappa$ such that $g_\alpha(x_0)\in A_n$, $g_\beta(x_0)\in B_m$. It follows from $A_n\cap B_m=\emptyset$ that $g_\alpha(x_0)\neq g_\beta(x_0)$. The property (1) of Definition 3.2 guarantees that $\gamma\neq\delta$. Without loss of generality, $\delta>\gamma$. Since $g_\beta(x_0)\neq g_\alpha(x_\gamma)$, the property (2) of Definition 3.2 guarantees that $g_\beta(x_\delta)\neq g_\alpha(x_\gamma)$ and this is the desired contradiction showing that $A_\omega\cap B_\omega=\emptyset$.

Now let us show that the set A_ω is open in (X,τ_φ). Given an ordinal $\alpha<\kappa$ with $g_\alpha(x_0)\in A_\omega$ we should find a set $F\in\varphi$ with $g_\alpha(F)\subset A_\omega$. Let $n\in\omega$ be the smallest number such that $g_\alpha(x_0)\in A_n$. We claim that the set $F=\{x_0\}\cup\{x\in X_{>\alpha} : g_\alpha(x)\notin B_n\}$ belongs to the filter φ. Assuming that $F\notin\varphi$, we conclude that the set $X_{>\alpha}\setminus F$ belongs to the family φ^+. Then for $k=n$ the set $E_k=\{x\in X_{>\alpha} : g_\alpha(x)\in B_k\}$ belongs to the family φ^+. Let $k\le n$ be the smallest number such that $E_k\in\varphi^+$.

We claim that $k>0$. Indeed, since B_0 is a closed subset in (X,τ_φ), its complement $X\setminus B_0$ is an open neighborhood of the point $g_\alpha(x_0)\in A_n$. Then the definition of the topology τ_φ yields a set $F_0\in\varphi$ such that $g_\alpha(F_0)\subset X\setminus B_0$. Since F_0 intersects E_k and is disjoint with E_0, we conclude that $k>0$.

Since $\varphi^+\not\ni E_{k-1}\subset E_k\in\varphi^+$, the set $E_k\setminus E_{k-1}$ is not empty and hence contains some point x_γ with $\gamma>\alpha$. Then $g_\alpha(x_\gamma)\in B_k\setminus B_{k-1}$ and hence $g_\alpha(x_\gamma)=g_\beta(x_\delta)$ for some ordinals $\beta<\delta<\kappa$ with $g_\beta(x_0)\in B_{k-1}$.

By the condition (1) of Definition 3.2, $\delta\neq\gamma$ (as $g_\alpha(x_\gamma)=g_\beta(x_\delta)$ and $g_\alpha(x_0)\neq g_\beta(x_0)$). If $\delta>\gamma$, then the equality $g_\alpha(x_\gamma)=g_\beta(x_\delta)$ is forbidden by the condition (2) of Definition 3.2 as $B_{k-1}\not\ni g_\alpha(x_\gamma)\neq g_\beta(x_0)\in B_{k-1}$. If $\gamma>\delta$, then the equality $g_\alpha(x_\gamma)=g_\beta(x_\delta)$ also is forbidden by (2) because $A_n\ni g_\alpha(x_0)\neq g_\beta(x_\delta)\in B_k$. The obtained contradiction shows that $F\in\varphi$ and $g_\alpha(F)\subset A_{n+1}\subset A_\omega$, witnessing that the set A_ω is open.

By analogy we can prove that the set B_ω is open in (X,τ_φ). Since A_ω and B_ω are disjoint open neighborhoods of the closed sets A_0,B_0, the topological T_1-space (X,τ_φ) is normal. □

The definition of the topology τ_φ implies that for every set $F\in\varphi$ the set $G(F)=\{g(x):g\in G,\ x\in F\}$ is closed-and-open in (X,τ_φ) and $X\setminus G(F)$ is discrete in (X,τ_φ).

Claim 3.7. $\{F\cap X_0 : F\in\varphi\}=\{U\cap X_0 : x_0\in U\in\tau_\varphi\}$ *and hence* $\chi(\varphi)\le\chi(x_0,\tau_\varphi)$.

Proof. The definition of the topology τ_φ guarantees that $\{U\in\tau_\varphi : x_0\in U\}\subset\varphi$ and hence $\{U\cap X_0 : x_0\in U\in\tau_\varphi\}\subset\{F\cap X_0 : F\in\varphi\}=\{F\in\varphi : F\subset X_0\}$.

To prove the reverse inclusion, fix any subset $F\in\varphi$ with $F\subset X_0$ and consider the set $U=F\cup(X\setminus X_0)$. We claim that $U\in\tau_\varphi$. Given any ordinal $\alpha<\kappa$ with $g_\alpha(x_0)\in U$, we need to find a set $E\in\varphi$ with $g_\alpha(E)\subset U$. Find $\beta<\kappa$ such that $g_\beta=\mathrm{id}_X$ and consider the set
$$E=\{x_0\}\cup\{x_\gamma\in F : \max\{\alpha,\beta\}<\gamma<\kappa\}\in\varphi.$$
We claim that $g_\alpha(E)\subset U$. Assuming the converse, we could find an ordinal $\gamma>\max\{\alpha,\beta\}$ such that $x_\gamma\in F$ and $g_\alpha(x_\gamma)\in X_0\setminus F$. Then $g_\alpha(x_\gamma)=x_\delta=g_\beta(x_\delta)$ for some ordinal $\delta<\kappa$. Since $x_\gamma\in F$ and $x_\delta\notin F$, the ordinals γ and δ are distinct.

If $\gamma<\delta$, then the inequality $g_\beta(x_0)=x_0\neq x_\delta=g_\alpha(x_\gamma)$ and the condition (2) of Definition 3.2 guarantee that $g_\beta(x_\delta)\neq g_\alpha(x_\gamma)$, which is a contradiction.

If $\gamma>\delta$, then the inequality $g_\alpha(x_0)\neq x_\delta=g_\beta(x_\delta)$ and the condition (2) of Definition 3.2 imply that $g_\alpha(x_\gamma)\neq g_\beta(x_\delta)=x_\delta$, which again leads to a contradiction. □

Claim 3.8. *The topology* τ_φ *has pseudocharacter* $\psi(x_0,\tau_\varphi)=\psi(\varphi)$ *at the point* x_0.

Proof. The inequality $\psi(\varphi)\le\psi(x_0,\tau_\varphi)$ follows from Claim 3.7. To show that $\psi(x_0,\tau_\varphi)\le\psi(\varphi)$, fix a subfamily $\mathcal F\subset\varphi$ such that $|\mathcal F|=\psi(\varphi)$ and $\cap\mathcal F=\{x_0\}$. For every $F\in\mathcal F$ define an open neighborhood $U^F\in\tau_\varphi$ of x_0 as the union $U^F=\bigcup_{n\in\omega}U^F_n$ of the sequence of sets $(U^F_n)_{n\in\omega}$ defined by the recursive formula: $U^F_0=\{x_0\}$ and
$$U^F_{n+1}=U^F_n\cup\{g_\alpha(x_\beta):\alpha<\beta<\kappa,\ x_\beta\in F,\ \ g_\alpha(x_0)\in U^F_n\}\ \ \text{for every } n\in\omega.$$
The definition of the topology τ_φ implies that $U^F=\bigcup_{n\in\omega}U^F_n$ is an open neighborhood of the point x_0 in X.

Let us show that $\bigcap_{F\in\mathcal F}U^F=\{x_0\}$. Assume conversely that this intersection contains a point x, distinct from x_0. For every $F\in\mathcal F$ find the smallest number $n_F\in\omega$ such that $x\in U^F_{n_F}$. Since $U^F_0=\{x_0\}\neq\{x\}$, we

conclude that $n_F > 0$ and hence $x \notin U^F_{n_F-1}$. By the definition of the set U^F_n, there are ordinals $\alpha_F < \beta_F < \kappa$ such that $x_{\beta_F} \in F$, $x = g_{\alpha_F}(x_{\beta_F}) \neq g_{\alpha_F}(x_0) \in U^F_{n_F-1}$.

Fix any set $F \in \mathcal{F}$. Since $x_{\beta_F} \in F$ and $x_{\beta_F} \notin \{x_0\} = \cap\mathcal{F}$, there is a set $E \in \mathcal{F}$ such that $x_{\beta_F} \notin E$. Then $\beta_F \neq \beta_E$. Without loss of generality, $\beta_F < \beta_E$. Since $\beta_E > \max\{\alpha_E, \beta_F, \alpha_F\}$ and $g_{\alpha_E}(x_0) \neq x = g_{\alpha_F}(x_{\beta_F})$, the condition (2) of Definition 3.2 guarantees that $x = g_{\alpha_E}(x_{\beta_E}) \neq g_{\alpha_F}(x_{\beta_F}) = x$, which is a desired contradiction that proves the equality $\bigcap_{F\in\mathcal{F}} U^F = \{x_0\}$ and the upper bound $\psi(x_0, \tau_\varphi) \le \psi(\varphi)$. □

□

4. Zariski G-topology and the existence of special filters

In light of Theorem 3.4 it is important to detect G-acts X that possess special sequences and special filters.

Proposition 4.1. *Let G be a monoid, X be a G-act, $x_0 \in X$ be a point, and λ be an infinite cardinal.*

(1) *If $|G| \le \kappa \le \psi(x_0, \zeta_G)$, then the G-act X contains a special sequence $(x_\alpha)_{\alpha<\kappa}$.*
(2) *If the G-act X contains a special sequence $(x_\alpha)_{\alpha<\kappa}$, then $|G| \le \kappa$ and $\mathrm{cf}(\kappa) \le \psi(x_0, \zeta_G)$.*

Proof. 1. Assume that $|G| \le \kappa \le \psi(x_0, \zeta_G)$ and let $G = \{g_\alpha : \alpha < \kappa\}$ be an enumeration of the monoid G such that $g_0 = 1_G$. By induction we shall construct an injective transfinite sequence $(x_\alpha)_{\alpha<\kappa}$ of points of the set X such that for any $\alpha < \kappa$ and $\beta, \gamma, \delta < \alpha$

(3) if $g_\beta(x_0) \neq g_\gamma(x_0)$, then $g_\beta(x_\alpha) \neq g_\gamma(x_\alpha)$;
(4) if $g_\beta(x_0) \neq g_\gamma(x_\delta)$, then $g_\beta(x_\alpha) \neq g_\gamma(x_\delta)$.

Assume that for some ordinal $\alpha < \kappa$ the points x_β, $\beta < \alpha$, have been constructed. For any ordinals $\beta, \gamma, \delta < \alpha$ consider the open neighborhoods

$$U_{\beta,\gamma} = \{x \in X : g_\beta(x_0) \neq g_\gamma(x_0) \ \Rightarrow g_\beta(x) \neq g_\gamma(x)\}$$

and

$$V_{\beta,\gamma,\delta} = \{x \in X : g_\beta(x_0) \neq g_\gamma(x_\delta) \ \Rightarrow \ g_\beta(x) \neq g_\gamma(x_\delta)\}$$

of x_0 in the Zariski G-topology ζ_G. Since $\psi(x_0, \zeta_G) \ge \kappa$, the intersection $\bigcap_{\beta,\gamma,\delta<\alpha} U_{\beta,\gamma} \cap V_{\beta,\gamma,\delta}$ has cardinality $\ge \kappa$ and hence contains some point $x_\alpha \in X \setminus \{x_\beta : \beta < \alpha\}$. It is clear that this point x_α satisfies the conditions (3), (4).

2. Now assume that the G-act X contains a special sequence $X_0 = \{x_\alpha\}_{\alpha<\kappa}$ for some infinite cardinal κ. Let $G = \{g_\alpha\}_{\alpha<\kappa}$ be an enumeration of the monoid G such that the conditions (1),(2) of Definition 3.2 are satisfied. Then $|G| \le \kappa$. We claim that $\psi(x_0, \zeta_G) \ge \mathrm{cf}(\kappa)$. Assuming the opposite, we can find a subfamily $\mathcal{U} \subset \tilde{\zeta}_G$ such that $\cap\mathcal{U} = \{x_0\}$ and $|\mathcal{U}| < \mathrm{cf}(\kappa)$. For each set $U \in \mathcal{U} \subset \tilde{\zeta}_G$ we can choose ordinals $\alpha_U, \beta_U < \kappa$ and a point $c_U \in X$ such that U is equal either to $\{x \in X : g_{\alpha_U}(x) \neq g_{\beta_U}(x)\}$ or to $\{x \in X : g_{\alpha_U}(x) \neq c_U\}$. If $c_U \in G(X_0)$, then we can find ordinals $\gamma_U, \delta_U < \kappa$ such that $c_U = g_{\gamma_U}(x_{\delta_U})$. In the opposite case, put $\gamma_U = \delta_U = 0$.

Since the set $A_U = \{\alpha_U, \beta_U, \gamma_U, \delta_U : U \in \mathcal{U}\}$ has cardinality $< \mathrm{cf}(\kappa)$, there is an ordinal $\alpha < \kappa$ such that $\alpha > \sup A_U$. We claim that $x_\alpha \in \cap\mathcal{U}$. To prove this inclusion, take any set $U \in \mathcal{U}$. If $U = \{x \in X : g_{\alpha_U}(x) \neq g_{\beta_U}(x)\}$, then the inclusion $x_0 \in U$ and the condition (1) of Definition 3.2 guarantee that $x_\alpha \in U$. If $U = \{x \in X : g_{\alpha_U}(x) \neq c_U\}$ and $c_U \in G(X_0)$, then the equality $c_U = g_{\gamma_U}(x_{\delta_U})$, the inclusion $x_0 \in U$ and the condition (2) of Definition 3.2 imply that $x_\alpha \in U$. If $c_U \notin G(X_0)$, then $g_{\alpha_U}(x_\alpha) \neq c_U$ and hence $x_\alpha \in U$. Therefore $x_\alpha \in \cap\mathcal{U} = \{x_0\}$, which is a desired contradiction. □

5. G-topologizability of G-acts

In this section we apply the results of the preceding sections and prove our main result:

Theorem 5.1. *Let G be a monoid and X be a G-act. If $|G| \le \kappa \le \psi(x_0, \zeta_G)$ for some point $x_0 \in X$ and some infinite cardinal κ, then for any infinite cardinal $\lambda \le \mathrm{cf}(\kappa)$ the G-act X admits 2^{2^κ} normal G-topologies with pseudocharacter λ at the point x_0.*

Proof. By Proposition 4.1, the space X contains a special sequence $X_0 = \{x_\alpha\}_{\alpha<\kappa}$. Let φ_0 be the filter on X generated by the sets $\{x_0\} \cup \{x_\beta : \beta > \alpha\}$, $\alpha < \kappa$. Denote by $\uparrow\varphi_0$ the set of all filters φ on X that contain the filter φ_0 and have $\cap\varphi = \{x_0\}$.

Claim 5.2. *For any infinite cardinal $\lambda \le \mathrm{cf}(\kappa)$ the set $\mathcal{F}_\lambda = \{\varphi \in \uparrow\varphi_0 : \psi(\varphi) = \lambda\}$ has cardinality $|\mathcal{F}| = 2^{2^\kappa}$.*

Proof. First observe that the family of all filters on the set X_0 has cardinality $\le 2^{2^\kappa}$. So, $|\mathcal{F}_\lambda| \le 2^{2^\kappa}$. To prove the reverse inequality, we consider two cases.

1. $\lambda = \mathrm{cf}(\kappa)$. Write the set X_0 as the disjoint union $X_0 = X_0' \cup X_0''$ of two sets of cardinality $|X_0'| = |X_0''| = \kappa$ such that $x_0 \in X_0'$. On the set X_0' consider the filter $\varphi_0|X_0' = \{F \cap X_0' : F \in \varphi_0\}$. The Pospíšil Theorem [11] (see also [6]) implies that the family $\mathcal{U}_0$ of all ultrafilters on X_0'' that contain the filter $\varphi_0|X_0'' = \{F \cap X_0'' : F \in \varphi_0\}$ has cardinality 2^{2^κ}. For any ultrafilter $u \in \mathcal{U}_0$ consider the filter $\varphi_u = \{A \subset X : A \cap X_0'' \in u,\ A \cap X_0' \in \varphi_0|X_0'\}$ and observe that $\psi(\varphi_u) = \psi(\varphi_0|X_0') = \mathrm{cf}(\kappa)$. Since for distinct ultrafilters $u, v \in \mathcal{U}_0$ the filters φ_u, φ_v are distinct, we conclude that $|\mathcal{F}_{\mathrm{cf}(\kappa)}| \ge 2^{2^\kappa}$.

2. $\lambda < \mathrm{cf}(\kappa)$. In this case the ordinal κ can be identified with the product $\kappa \times \lambda$ endowed with the lexicographic order: $(\alpha, \beta) < (\alpha, \beta')$ iff $\alpha < \alpha'$ or ($\alpha = \alpha'$ and $\beta < \beta'$). Let $\xi : \kappa \times \lambda \to \kappa$ be the order isomorphism. On the cardinal λ consider the filter φ_λ of cofinite subsets. This filter has pseudocharacter $\psi(\varphi_\lambda) = \lambda$. By the preceding case, the family $\mathcal{F}_{\mathrm{cf}(\kappa)}$ has cardinality 2^{2^κ}. For any filter $u \in \mathcal{F}_{\mathrm{cf}(\kappa)}$ consider the filter φ_u on X generated by the sets

$$\Phi_{U,L} = \{x_0\} \cup \{x_{\xi(\alpha,\beta)} : x_\alpha \in U,\ \beta \in L\} \text{ where } U \in u,\ L \in \varphi_\lambda.$$

It can be shown that $\psi(\varphi_u) = \psi(\varphi_\lambda) = \lambda$ and for distinct filters $u, v \in \mathcal{F}_{\mathrm{cf}(\kappa)}$ the filters φ_u and φ_v are distinct. Consequently, $|\mathcal{F}_\lambda| \ge |\mathcal{F}_{\mathrm{cf}(\kappa)}| \ge 2^{2^\kappa}$. □

For any filter $\varphi \in \mathcal{F}_\lambda$ the G-topology τ_φ on X is normal and has pseudocharacter $\psi(x_0, \tau_\varphi) = \psi(\varphi) = \lambda$ at x_0 according to Theorem 3.4. Theorem 3.4(3) implies that for distinct filters $u, v \in \mathcal{F}_\lambda$ the topologies τ_u and τ_v are distinct. Consequently, X admits at least $|\mathcal{F}_\lambda| = 2^{2^\kappa}$ normal G-topologies with pseudocharacter λ at x_0. □

Remark 5.3. Example 2.9 shows that Theorem 5.1 cannot be reversed: the group $G = H^\kappa$ is normally G_s-topologizable but $\psi(x_0, \zeta_{G_s}) = \kappa < 2^\kappa = |G_s|$ for any point x_0.

For countable monoids G, Theorem 5.4 implies the following characterization of G-topologizability that answers Problem 1.2.

Theorem 5.4. *For countable monoid G and a G-act X the following conditions are equivalent:*

(1) *X admits a non-discrete Hausdorff G-topology;*
(2) *the Zariski G-topology ζ_G on X is not discrete;*
(3) *X admits $2^{\mathfrak{c}}$ non-discrete normal G-topologies.*

We do not know if this theorem holds for arbitrary G-acts.

Problem 5.5. *Let G be an uncountable monoid (group). Is a G-act X G-topologizable if its Zariski G-topology ζ_G is not discrete?*

It may happen that the results of [5] can help to give an answer to this problem.

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